

CONVERGENCE OF EIGENVALUES TO THE SUPPORT OF THE LIMITING MEASURE IN CRITICAL β MATRIX MODELS

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ABSTRACT. We consider the convergence of the eigenvalues to the support of the equilibrium measure in the β matrix models at criticality. We show a phase transition phenomenon, namely that, with probability one, all eigenvalues will fall in the support of the limiting spectral measure when $\beta > 1$, whereas this fails when $\beta < 1$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

1.1. Definitions and Known Results. Let B be a subset of the real line. B can be chosen as the whole real line, an interval, or the union of finitely many disjoint intervals. For now, let $V : B \rightarrow \mathbb{R}$ be an arbitrary function, and let $\beta > 0$ be a positive real number. In this paper, we consider the β ensemble, i.e a sequence of N random variables $(\lambda_1, \dots, \lambda_N)$ with law $\mu_{N,\beta}^{V;B}$ defined as the probability measure on B^N given by

$$d\mu_{N,\beta}^{V;B}(\lambda) = \frac{1}{Z_{N,\beta}^{V;B}} \prod_{i=1}^N d\lambda_i e^{-\frac{N\beta}{2}V(\lambda_i)} \mathbf{1}_B(\lambda_i) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta, \quad (1.1)$$

where $Z_{N,\beta}^{V;B}$ is the partition function

$$Z_{N,\beta}^{V;B} = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{i=1}^N d\lambda_i e^{-\frac{N\beta}{2}V(\lambda_i)} \mathbf{1}_B(\lambda_i) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta. \quad (1.2)$$

If β is equal to 1, 2, or 4, $\mu_{N,\beta}^{V;\mathbb{R}}$ is the probability measure induced on the eigenvalues of Ω by the probability measure $d\Omega e^{-\frac{N\beta}{2}\text{Tr}(V(\Omega))}$ on a vector space of real symmetric, Hermitian, and self-dual quaternionic $N \times N$ matrices respectively, see [Meh04].

Therefore, the β models can be viewed as the natural generalization of these matrix models and we will refer to λ_i as “eigenvalue” of a “matrix model”. For a quadratic potential, the β ensembles can also be realized as the eigenvalues

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of tridiagonal matrices [DE02]. Even though such a construction is not known for general potentials, β matrix models are natural Coulomb interaction probability measures which appear in many different settings. These laws have been intensively studied, both in physics and in mathematics. In particular, the convergence of the empirical measure of the λ_i 's (which we will call hereafter the spectral measure) was proved [ST97, Dei99, AGZ10], and its fluctuations analyzed [Joh98, Pas06, Shc13]. Moreover, the partition functions as well as the mean Stieltjes transforms can be expanded as a function of the dimension to all orders [BIPZ78, ACKM93, ACM92, Ake96, CE06, Che06, BG11, BG13]. It turns out that both central limit theorems and all order expansions depend heavily on whether the limiting spectral measure has a connected support. Indeed, when the limiting spectral measure has a disconnected support, it turns out that even though most eigenvalues will stick into one of its connected components, some eigenvalues will randomly switch from one to the other connected components of the support even at the large dimension limit. This phenomenon can invalidate the central limit theorem, see e.g. [Pas06, Shc13], and results with the presence of a Theta function in the large dimension expansion of the partition function [BG13]. In the case where the limiting measure has a connected support \mathbf{S} , and that the eigenvalues are assumed to belong asymptotically to \mathbf{S} , even more refined information could be derived. Indeed, in this case, local fluctuations of the λ_i 's could first be established in the case corresponding to Gaussian random matrices, $\beta = 1, 2$ or 4 and $V(x) = x^2$ [Meh04], then to tridiagonal ensembles (all $\beta \geq 0$ but $V(x) = x^2$) [RRV06] and more recently for general potentials and $\beta \geq 0$ [BEY12, BEY, ?, Shc13, FB]. However, all these articles consider non-critical potentials. We shall below define more precisely the later case but let us say already that a non-critical potential prevents the eigenvalues to deviate from the support of the limiting spectral measure as the dimension goes to infinity. We study in this article β models with critical potentials and whether the eigenvalues stay confined in the limiting support. In fact, we exhibit an interesting phase transition: we show that if $\beta > 1$ the eigenvalues stay confined whereas if $\beta < 1$ some deviate towards the critical point with probability one. We postpone the study of the critical case $\beta = 1$ to further research. Let us finally point out that the case where the potential is critical, but with critical parameters tuned with the dimension so that new phenomena occur, was studied in [Cla, Eyn06]. We restrict ourselves to potentials independent of the dimension.

We next describe more precisely the definition of criticality and our results.

Consider the *spectral measure* $L_N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$, where δ_{λ_i} is the Dirac measure centered on λ_i . L_N belongs to the set $M_1(\mathbf{B})$ of probability measures on the real line. We endow this space with the weak topology. Then, L_N converges almost surely. This convergence can be derived from the following large deviation result (see [BAG97], and [AGZ10, Theorem 2.6.1]) :

Theorem 1.1. *Assume that V is continuous and goes to infinity faster than $2 \log |x|$ (if \mathbf{B} is not bounded). The law of L_N under $\mu_{N,\beta}^{V;\mathbf{B}}$ satisfies a large deviation principle with speed N^2 and good rate function $\tilde{\mathcal{E}}$, where $\tilde{\mathcal{E}} = \mathcal{E} - \inf\{\mathcal{E}(\mu), \mu \in M_1(\mathbf{B})\}$ with*

$$\mathcal{E}[\mu] = \frac{\beta}{4} \iint (V(\xi) + V(\eta) - 2 \log |\xi - \eta|) d\mu(\xi) d\mu(\eta). \quad (1.3)$$

In other words,

(1)

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_{N,\beta}^{V;\mathbf{B}} = - \inf_{\mu \in M_1(\mathbf{B})} \mathcal{E}[\mu].$$

(2) $\tilde{\mathcal{E}} : M_1(\mathbb{R}) \rightarrow [0, \infty]$ possesses compact level sets $\{v : \tilde{\mathcal{E}}(v) \leq M\}$ for all $M \in \mathbb{R}^+$.

(3) for any open set $O \subset M_1(\mathbf{B})$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mu_{N,\beta}^{V;\mathbf{B}}(L_N \in O) \geq - \inf_O \tilde{\mathcal{E}}.$$

(4) for any closed set $F \subset M_1(\mathbf{B})$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mu_{N,\beta}^{V;\mathbf{B}}(L_N \in F) \leq - \inf_F \tilde{\mathcal{E}}.$$

The minimizers of \mathcal{E} are described as follows (see [AGZ10, Lemma 2.6.2]):

Theorem 1.2. \mathcal{E} achieves its minimal value at a unique minimizer μ_{eq} . Moreover, μ_{eq} has a compact support \mathbf{S} . In addition, there exists a constant C_V such that:

$$\begin{cases} \text{for } x \in \mathbf{S} & 2 \int_{\mathbb{R}} d\mu_{\text{eq}}(\xi) \ln |x - \xi| - V(x) = C_V \\ \text{for } x \text{ Lebesgue almost everywhere in } \mathbf{S}^c & 2 \int_{\mathbb{R}} d\mu_{\text{eq}}(\xi) \ln |x - \xi| - V(x) < C_V. \end{cases} \quad (1.4)$$

We will refer to μ_{eq} , which is compactly supported, as the equilibrium measure.

Remark 1.3. Theorem 1.1 and Theorem 1.2 imply that under $\mu_{N,\beta}^{V;\mathbf{B}}$, L_N converges to the equilibrium measure μ_{eq} almost surely.

Once the existence of the equilibrium measure is established, one may explore the convergence of the eigenvalues to the support of the equilibrium measure μ_{eq} . It is shown in [BG11, BG13] that the probability that eigenvalues escape this limiting support is governed by a large deviation principle with rate function given by

$$\tilde{\mathcal{J}}^{V;\mathbf{B}}(x) = \mathcal{J}^{V;\mathbf{B}}(x) + C_V \quad (1.5)$$

with

$$\mathcal{J}^{V;\mathbf{B}}(x) = \begin{cases} V(x) - 2 \int d\mu_{\text{eq}}(\xi) \ln |x - \xi| & x \in \mathbf{B} \setminus \mathbf{S} \\ -C_V & \text{otherwise.} \end{cases} \quad (1.6)$$

The large deviation principle states as follows:

Theorem 1.4. Assume V continuous and going to infinity faster than $2 \log |x|$ (in the case where \mathbf{B} is not bounded). Then

(1) $\tilde{\mathcal{J}}^{V;\mathbf{B}}$ is a good rate function.

(2) We have large deviation estimates: for any $F \subseteq \overline{\mathbf{B} \setminus \mathbf{S}}$ closed and $O \subseteq \mathbf{B} \setminus \mathbf{S}$ open,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;\mathbf{B}}[\exists i \quad \lambda_i \in F] &\leq -\frac{\beta}{2} \inf_{x \in F} \tilde{\mathcal{J}}^{V;\mathbf{B}}(x), \\ \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;\mathbf{B}}[\exists i \quad \lambda_i \in O] &\geq -\frac{\beta}{2} \inf_{x \in O} \tilde{\mathcal{J}}^{V;\mathbf{B}}(x). \end{aligned}$$

The last theorem shows that the support of the spectrum is governed by the minimizers of $\tilde{\mathcal{J}}^{V;\mathbf{B}}$.

Definition 1.5. Assume V is continuous. We say that V is non-critical iff $\tilde{\mathcal{J}}^{V;\mathbf{B}}$ is positive everywhere outside of the support of μ_{eq} .

Remark 1.6. Theorem 1.2 only ensures $\tilde{\mathcal{J}}^{V;\mathbf{B}}$ is positive **almost** everywhere outside of the support of μ_{eq} .

Remark 1.7. In the literature, the potential V is also said to be critical when the density of the equilibrium measure vanishes at an interior point or at an edge point at a fast rate. We will assume that our potential is not critical in this sense (see the fourth point in Assumption 1).

A consequence of the second part of the aforementioned theorem is the following:

Corollary 1.8. Let the assumptions in Theorem 1.4 hold. Assume that V is non-critical. Then

$$\lim_{N \rightarrow \infty} \mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) = 0, \quad (1.7)$$

for any open set \mathbf{A} containing the support of μ_{eq} .

Remark 1.9. Since the law of the eigenvalues satisfies a large deviation principle with rate N , the eigenvalues actually converge to the support exponentially fast (or more precisely, $\exists c > 0$, s.t. $\mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) \leq e^{-cN}$ for any open set \mathbf{A} containing the support of μ_{eq}).

Remark 1.10. By the definition of the partition function, $1 - \mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) = \frac{Z_{N,\beta}^{V;\mathbf{A}}}{Z_{N,\beta}^{V;\mathbf{B}}}$, thus,

$$(1.7) \Leftrightarrow \lim_{N \rightarrow \infty} \frac{Z_{N,\beta}^{V;\mathbf{A}}}{Z_{N,\beta}^{V;\mathbf{B}}} = 1. \quad (1.8)$$

In the rest of this article we investigate what happens in the case where V is critical. This investigation will require the uses of precise estimates on β models partitions functions derived in [BG11, BG13] and to apply these results we shall make the following assumption :

Assumption 1. • $V : \mathbf{B} \rightarrow \mathbb{R}$ is a continuous function independent of N .
• If $\pm\infty \in \mathbf{B}$,

$$\liminf_{x \rightarrow \pm\infty} \frac{V(x)}{2 \ln |x|} > 1. \quad (1.9)$$

- $\text{supp}(\mu_{\text{eq}})$ is a finite union of disjoint intervals, i.e. $\text{supp}(\mu_{\text{eq}})$ of the form $\mathbf{S} = \bigcup_{h=1}^g \mathbf{S}_h$, where $\mathbf{S}_h = [\alpha_h^-, \alpha_h^+]$.
- Let $\mathbf{B} = \bigcup_h [b_h^-, b_h^+]$ with $b_h^- \leq \alpha_h^- \leq \alpha_h^+ \leq b_h^+$ and set $\text{Hard} = \{a \in \bigcup \{\alpha_h^-, \alpha_h^+\} : \alpha_h^\pm = b_h^\pm\}$ and $\text{Soft} = \bigcup \{\alpha_h^-, \alpha_h^+\} \setminus \text{Hard}$. Then we assume that

$$S(x) = \pi \frac{d\mu_{\text{eq}}}{dx} \sqrt{\left| \frac{\prod_{a \in \text{Hard}} (x-a)}{\prod_{a \in \text{Soft}} (x-a)} \right|}.$$

is strictly positive whenever $x \in \mathbf{S}$.

- V is a real analytic function in some open neighborhood \mathbf{A} of \mathbf{S} : $\mathbf{A} = \bigcup_{h=1}^g \mathbf{A}_h$, $\mathbf{A}_h = (a_h^-, a_h^+)$ for some $a_h^- < \alpha_h^- < \alpha_h^+ < a_h^+$, and $\mathbf{A}_h, \mathbf{A}_{h'}$ are disjoint for any $h \neq h'$.

Remark 1.11. When V is analytic in a neighborhood of the real line, the third point of our assumption is automatically satisfied. Here, we assume analyticity only in a neighborhood of \mathbf{S} .

Remark 1.12. Hereafter the neighborhood \mathbf{A} will be fixed, but clearly can be chosen as small as wished, being given it is open and containing \mathbf{S} .

We want to investigate whether (1.7) still holds when the restriction on $\tilde{\mathcal{J}}^{V;\mathbf{B}}$ is weakened so that it vanishes outside the support \mathbf{S} . Our working hypothesis will be the following:

Assumption 2. With \mathbf{A} given in Assumption 1, assume that $\tilde{\mathcal{J}}^{V;\mathbf{B}}$ vanishes only on the support of the equilibrium measure \mathbf{S} and at one point c_0 in $\overline{\mathbf{A}}^c$. We also require that V (and therefore $\tilde{\mathcal{J}}^{V;\mathbf{B}}$) extends as a twice continuously differentiable function in a open neighborhood $(c_0 - \varepsilon, c_0 + \varepsilon)$ of c_0 , for some $\varepsilon > 0$ and that $\frac{d^2}{dx^2}\tilde{\mathcal{J}}^{V;\mathbf{B}}(c_0) > 0$. Moreover, for technical reason, we require $\frac{d^2}{dx^2}V \geq \sigma > 0$ on \mathbf{A} .

Remark 1.13. Because $\tilde{\mathcal{J}}^{V;\mathbf{B}}$ is a good rate function which is positive outside $\mathbf{A} \cup (c_0 - \varepsilon, c_0 + \varepsilon)$ for any $\varepsilon > 0$ small enough under our assumptions, Theorem 1.4 implies that $\mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A} \cup (c_0 - \varepsilon, c_0 + \varepsilon))$ goes to zero exponentially fast. In other words, it is bounded above by e^{-Nc_ε} for some $c_\varepsilon > 0$.

Potentials V satisfying Assumptions 1 and 2 are easy to build. Indeed, being given a probability measure μ so that $x \rightarrow f(x) = 2 \int \log|x-y|d\mu(y)$ is well defined and continuous on the whole real line, we simply choose V to be equal to f on the support \mathbf{S} of μ and Lebesgue almost surely strictly greater than f outside \mathbf{S} . This insures that μ is the equilibrium measure of our β -model with potential V as it satisfies (1.4). We can then choose $V - f$ strictly positive outside \mathbf{S} except at the point c_0 where it is strictly convex. To make sure that V also satisfies Assumption 1, we can take μ to be the equilibrium measure for a potential W which is real-analytic, going to infinity faster than $2 \ln|x|$, strictly convex in a neighborhood of \mathbf{S} , and non-critical (in the one cut case, we can take W strictly convex everywhere). We let $\mathbf{A} = \cup_{h=1}^g(a_h^-, a_h^+)$ be a bounded, open neighborhood of the support of μ . We choose $V = W + C_W$ on $(-\infty, a_g^+)$. We may assume without loss of generality that $\tilde{\mathcal{J}}^{V;\mathbf{B}}(a_g^+)$ is strictly positive. We then choose for $x > a_g^+$, $V(x) - f(x) = d(x - c_0)^2$ for some $d > 0$ and $c_0 > a_g^+$ so that $d(a_g^+ - c_0)^2 = \tilde{\mathcal{J}}^{W;\mathbf{B}}(a_g^+)$. We have constructed an equilibrium measure μ for a potential V satisfying Assumptions 1 and 2.

1.2. Main Results.

Theorem 1.14. Given Assumptions 1 and 2, and with \mathbf{A} as in assumption 1, c_0, ε as in Assumption 2, we have the following alternative :

- when $\beta > 1$,

$$\lim_{N \rightarrow \infty} \mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) = 0, \quad (1.10)$$

- when $\beta < 1$,

$$\lim_{N \rightarrow \infty} \mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) = 1. \quad (1.11)$$

Equivalently, for any $\varepsilon \in (0, \varepsilon)$, the probability that there exists an eigenvalue in $(c_0 - \varepsilon, c_0 + \varepsilon)$ goes to zero when $\beta > 1$ and to one when $\beta < 1$.

The behavior below $\beta = 1$ can be illustrated with the case $\beta = 0$ where one would consider a potential V vanishing on a support \mathbf{S} and at a point c_0 (where its second derivative is positive), being strictly positive everywhere else. This corresponds to N independent variables with probability of order $N^{-1/2}$ to belong to a small neighborhood of c_0 (where the latter probability can be estimated by Laplace method). However, we will show in this article that this weight has to be corrected by a term $N^{-\frac{\beta}{2}}$ by studying the precise estimates derived in [BG11, BG13]. Essentially, digging back into the latter estimates, one can realize that these corrections come from Selberg integral and in fact is due to the Coulomb gas interaction. In this case, it is clear that some eigenvalues will lie in the neighborhood of c_0 with positive probability as soon as $\beta + 1 < 2$. The existence of a phase transition for this phenomenon at $\beta = 1$ is new to our knowledge. It suggests that the support of the eigenvalues of matrices with real coefficients corresponding to $\beta = 1$ matrix models might be more sensible to perturbations of the potential than matrices with complex coefficients (corresponding to $\beta = 2$). This is however not supported by finite dimensional perturbations of the Wigner matrices since the BBP transition [BBAP05] does not vary much between these two cases. Let us observe that our arguments could be carried similarly with several critical points similar to c_0 without changing the phase transition. However, if the second derivative of $\mathcal{J}^{V;\mathbf{B}}$ at these critical points could vanish so that $\mathcal{J}^{V;\mathbf{B}}$ behaves as $|x - c_0|^q$ in the vicinity of c_0 for some $q > 2$, the phase transition would occur at a threshold β_q depending on q (see Remark 5.4).

1.3. Structure of the paper. In Section 2 we reduce the problem to the analysis of the probability that M eigenvalues are contained in a small neighborhood of c_0 while the rest of the $N - M$ eigenvalues are contained in \mathbf{A} and state the main proposition, Proposition 2.1, which give precise estimates of this probability. We deduce our main result Theorem 1.14 in the case $\beta > 1$ in Section 3 and the case $\beta < 1$ in Section 4. Section 5 is devoted to the proof of Proposition 2.1, which we first give in the case where the equilibrium measure has a connected support and then extend to the general case. The appendix contains precise concentration of measures results which are key to our estimates.

1.4. Notation. We use the notation $X \lesssim Y$ (resp. $X \gtrsim Y$) to denote $X \leq CY$ (resp. $X \geq CY$) for some universal constant C . $X \approx Y$ when both $X \lesssim Y$ and $X \gtrsim Y$ hold. We sometimes use $a \ll 1$ to denote that a is smaller than any universal constant involved in the proof.

2. PRELIMINARY AND BASIC ANALYSIS

The probability that a specific subset of M eigenvalues are contained in a small neighborhood $(c_0 - \epsilon, c_0 + \epsilon)$ of c_0 while the other $N - M$ eigenvalues are contained in \mathbf{A} shall be denoted by $\Phi_{N,M,\beta}^{V;\epsilon}$:

$$\Phi_{N,M,\beta}^{V;\epsilon} := \mu_{N,\beta}^{V;\mathbf{B}}(\lambda_{N-M+1}, \dots, \lambda_N \in (c_0 - \epsilon, c_0 + \epsilon), \lambda_1, \dots, \lambda_{N-M} \in \mathbf{A}). \quad (2.1)$$

$\Phi_{N,M,\beta}^{V;\epsilon}$ depends also on \mathbf{A} and \mathbf{B} but it will be fixed hereafter as in Assumption 1 so that we do not stress this dependency. ϵ will later be chosen small enough, but notice that the conclusion will not depend on this choice since the probability that eigenvalues go in $[c_0 - \epsilon, c_0 + \epsilon] \setminus [c_0 - \epsilon, c_0 + \epsilon]$ goes to zero as N goes to infinity for any $\epsilon > 0$ by the large deviation principle Theorem 1.4 and Assumption 2. The

key to prove our main result is to compute the speed at which $\Phi_{N,M,\beta}^{V;\epsilon}$ goes to zero as N goes to $+\infty$. Indeed, note that

$$\begin{aligned} \mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) &= \mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A} \cup (c_0 - \epsilon, c_0 + \epsilon)) \\ &+ \sum_{M > \delta N} \binom{N}{M} \Phi_{N,M,\beta}^{V;\epsilon} \\ &+ \sum_{1 \leq M \leq \delta N} \binom{N}{M} \Phi_{N,M,\beta}^{V;\epsilon} \\ &=: P_1 + P_2 + P_3. \end{aligned} \quad (2.2)$$

Here $\delta > 0$ is a small fixed constant which will be chosen later.

Since $\tilde{\mathcal{J}}^{V;\mathbf{B}}$ is a good rate function which is positive outside $\mathbf{A} \cup [c_0 - \epsilon, c_0 + \epsilon]$, Theorem 1.4 implies that for any fixed $\epsilon > 0$, P_1 approaches 0 exponentially fast. In other words, it is controlled by e^{-Nc_ϵ} for some $c_\epsilon > 0$.

P_2 is bounded above by the probability that there are at least δN eigenvalues close to c_0 . This implies that the empirical measure L_N must put a mass δ in $(c_0 - \epsilon, c_0 + \epsilon)$, so that it must be at a positive distance of μ_{eq} since $(c_0 - \epsilon, c_0 + \epsilon) \cap \mathbf{S} = \emptyset$. By the large deviation principle for the law of L_N described in Theorem 1.1, P_2 is bounded above by $e^{-c_\delta N^2}$ for some $c_\delta > 0$. Therefore we deduce that for any $\delta, \epsilon > 0$, there exists $c(\delta, \epsilon) > 0$ such that

$$\mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) = P_3 + O(e^{-c(\delta, \epsilon)N}). \quad (2.3)$$

Our goal, therefore, is to control the third term P_3 .

Since $\mathcal{J}^{V;\mathbf{B}}$ goes to infinity at infinity, Theorem 1.4 also shows that the probability to have an eigenvalue above some finite threshold goes to zero exponentially fast. Therefore, we may assume without loss of generality that \mathbf{B} is a bounded set.

Thus, we are left to analyze $\Phi_{N,M,\beta}^{V;\epsilon}$ for a bounded set \mathbf{B} .

We prove the following bounds in section 5.2:

Proposition 2.1. *Let Assumptions 1 and 2 hold. Then, there exist $c, \delta_0, \epsilon_0 > 0$ so that for $\delta \in (0, \delta_0)$, $\epsilon \in (0, \epsilon_0 \wedge \epsilon)$, we have uniformly in $M \leq \delta N$*

$$\Phi_{N,M,\beta}^{V;\epsilon} \lesssim \frac{1}{N^{\frac{M(\beta+1)}{2}}} + O(e^{-cN^2}). \quad (2.4)$$

On the other hand,

$$\frac{1}{N^{\frac{(\beta+1)}{2}}} \frac{Z_{N,\beta}^{V;\mathbf{A}}}{Z_{N,\beta}^{V;\mathbf{B}}} \lesssim \Phi_{N,1,\beta}^{V;\epsilon} + O(e^{-cN^2}). \quad (2.5)$$

If one assumes \mathbf{A} is connected, Proposition 2.1 follows from the calculation from [BG11] where the one-cut case is considered. In the general multi-cut case, the proof of Proposition 2.1 is based on the precise estimate derived in [BG13] for the partition function and correlators for fixed filling fraction measure, that is with given number of eigenvalues in each connected part of the support \mathbf{S} . One could also consider the more general multi-cut case where the potential V is replaced by a non-linear statistic and then use similar estimates derived in [?].

We next give the proof of our main result.

3. CONVERGENCE OF THE EIGENVALUES TO THE SUPPORT \mathbf{S} WHEN $\beta > 1$

We next prove the first half of our main Theorem 1.14. To this end, we use the upper bound (2.4) provided by Proposition 2.1 to find

$$P_3 \leq \sum_{1 \leq M \leq \delta N} \binom{N}{M} \Phi_{N,M,\beta}^{V;\epsilon} \lesssim \left(1 + N^{-\frac{(\beta+1)}{2}}\right)^N - 1. \quad (3.1)$$

where we used that any error of order e^{-cN^2} in $\Phi_{N,M,\beta}^{V;\epsilon}$, $1 \leq M \leq \delta N$, is neglectable in the above sum. Hence, when $\beta > 1$, P_3 goes to 0 as N goes to $+\infty$. We deduce by (2.3) that

$$\lim_{N \rightarrow \infty} \mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \notin \mathbf{A}) = 0.$$

Moreover, this is equivalent to the fact that $\mu_{N,\beta}^{V;\mathbf{B}}(\exists \lambda_i \in (c_0 - \epsilon, c_0 + \epsilon))$ goes to zero by Remark 1.13.

4. ESCAPING EIGENVALUES WHEN $\beta < 1$

We prove that when $\beta < 1$ the probability that no eigenvalues lies in the neighborhood of c_0 goes to zero, that is by Remark 1.10, that we have:

$$\lim_{N \rightarrow \infty} \frac{Z_{N,\beta}^{V;\mathbf{A}}}{Z_{N,\beta}^{V;\mathbf{B}}} = 0. \quad (4.1)$$

This is done by lower bounding the probability $p_{N,\beta}^V$ that one eigenvalue exactly lies in the neighborhood of c_0 . Indeed, as $A_i = \{\lambda_i \in (c_0 - \epsilon, c_0 + \epsilon), \lambda_j \in \mathbf{A}, j \neq i\}$ are disjoint as soon as $(c_0 - \epsilon, c_0 + \epsilon) \cap \mathbf{A} = \emptyset$, we have by symmetry

$$p_{N,\beta}^V = N \Phi_{N,1,\beta}^{V;\epsilon}.$$

Since $p_{N,\beta}^V \leq 1$, we deduce from (2.5) that

$$\frac{N}{N^{\frac{(\beta+1)}{2}}} \frac{Z_{N,\beta}^{V;\mathbf{A}}}{Z_{N,\beta}^{V;\mathbf{B}}} \lesssim 1 + N \times O(e^{-cN^2}),$$

which results with

$$\frac{Z_{N,\beta}^{V;\mathbf{A}}}{Z_{N,\beta}^{V;\mathbf{B}}} \lesssim N^{\frac{\beta-1}{2}},$$

so that (4.1), and therefore the second part of our main Theorem 1.14, follows.

We first prove Proposition 2.1 in the case where μ_{eq} has a connected support (the one cut case) where our proof is based on the expansion of partition functions obtained in the one cut case in [BG11], and then turn to the more delicate general case (multi-cut case) which is based from estimates from [BG13].

5. PROOF OF PROPOSITION 2.1

We start with the explicit formula for $\Phi_{N,M,\beta}^{V;\epsilon}$ which can be written as follows :

$$\Phi_{N,M,\beta}^{V;\epsilon} = \frac{Z_{N-M,\beta}^{\frac{N}{N-M}V;\mathbf{A}}}{Z_{N,\beta}^{V;\mathbf{B}}} \int_{[c_0-\epsilon, c_0+\epsilon]^M} \Xi(\eta_1, \dots, \eta_M) \prod_{1 \leq k < l \leq M} |\eta_k - \eta_l|^\beta \prod_{j=1}^M e^{-\frac{\beta M}{2} V(\eta_j)} d\eta_j$$

where

$$\Xi(\eta_1, \dots, \eta_M) := \mu_{N-M, \beta}^{\frac{N}{N-M} V; \mathbf{A}} \left(\prod_{j=1}^M e^{\beta \sum_{i=1}^{N-M} \ln |\eta_j - \lambda_i| - \frac{\beta}{2} (N-M) V(\eta_j)} \right)$$

The term $\prod_{1 \leq k < l \leq M} |\eta_k - \eta_l|^\beta$ is bounded by $(2\epsilon)^{\frac{\beta M(M-1)}{2}}$. Thus, we have:

$$\Phi_{N,M,\beta}^{V;\epsilon} \leq (2\epsilon)^{\frac{\beta M(M-1)}{2}} Y_{N,M} L_{N,M}. \quad (5.1)$$

with

$$\begin{aligned} L_{N,M} &:= \int_{[c_0 - \epsilon, c_0 + \epsilon]^M} \Xi(\eta_1, \dots, \eta_M) \prod_{j=1}^M e^{-\frac{M\beta}{2} V(\eta_j)} d\eta_j. \\ Y_{N,M} &:= \frac{Z_{N-M,\beta}^{\frac{N}{N-M} V; \mathbf{A}}}{Z_{N,\beta}^{V; \mathbf{B}}}, \end{aligned} \quad (5.2)$$

We wish to split $Y_{N,M}$ into components for further analysis. We make the decomposition:

$$Y_{N,M} = \frac{Z_{N,\beta}^{V; \mathbf{A}}}{Z_{N,\beta}^{V; \mathbf{B}}} \tilde{Y}_{N,M}, \quad \tilde{Y}_{N,M} = F_{N,M} G_{N,M}, \quad (5.3)$$

where

$$F_{N,M} = \frac{Z_{N-M,\beta}^{\frac{N}{N-M} V; \mathbf{A}}}{Z_{N-M,\beta}^{V; \mathbf{A}}}, \quad G_{N,M} = \frac{Z_{N-M,\beta}^{V; \mathbf{A}}}{Z_{N,\beta}^{V; \mathbf{A}}}. \quad (5.4)$$

To get the upper bound in Proposition 2.1, we will use the trivial estimate $Y_{N,M} \leq \tilde{Y}_{N,M}$ since $Z_{N,\beta}^{V; \mathbf{A}} \leq Z_{N,\beta}^{V; \mathbf{B}}$. Thus one finally rewrites formula (5.1) as:

$$\Phi_{N,M,\beta}^{V;\epsilon} \leq (2\epsilon)^{\frac{\beta M(M-1)}{2}} G_{N,M} F_{N,M} L_{N,M}. \quad (5.5)$$

We remark here that when $M=1$, the inequality (5.1) becomes an equality:

$$\Phi_{N,1,\beta}^{V;\epsilon} = \frac{Z_{N,\beta}^{V; \mathbf{A}}}{Z_{N,\beta}^{V; \mathbf{B}}} G_{N,1} F_{N,1} L_{N,1}. \quad (5.6)$$

(5.6) will be used to prove (2.5).

We next estimate $G_{N,M}$ and $F_{N,M} \times L_{N,M}$. We shall prove that

Proposition 5.1. *Under Assumptions 1 and 2, there exists a small $\delta > 0$, such that uniformly for $M \leq \delta N$, we have*

$$G_{N,M} \approx C_M \frac{1}{N^{\frac{M\beta}{2}}} e^{NM(\frac{\beta}{2} \inf_{\xi \in \mathbf{B}} \mathcal{J}^{V; \mathbf{B}}(\xi) + \frac{\beta}{2} \int V(\eta) d\mu_{\text{eq}}(\eta))} \left(\frac{N-M}{N} \right)^{\frac{\beta}{2}(N-M)}. \quad (5.7)$$

Furthermore, there exists a finite positive constant C such that for all $1 \leq M \leq \delta N$,

$$\frac{1}{C} e^{-CM^2} \leq C_M \leq C e^{CM^2}. \quad (5.8)$$

For the term $F_{N,M} \times L_{N,M}$ we have the estimate

Proposition 5.2. *Under Assumptions 1 and 2, there exists a positive constant c and a finite constant C so that for $\delta > 0$ small enough, such that uniformly on $M \leq \delta N$,*

$$F_{N,M} L_{N,M} \lesssim e^{CM^2} e^{-\frac{\beta}{2} NM (\int V(\eta) d\mu_{\text{eq}}(\eta) + \inf \mathcal{J}^{V;B})} \frac{1}{N^{\frac{M}{2}}} + O(e^{-cN^2})$$

and

$$F_{N,1} L_{N,1} \gtrsim e^{-\frac{\beta}{2} N (\int V(\eta) d\mu_{\text{eq}}(\eta) + \inf \mathcal{J}^{V;B})} \frac{1}{N^{\frac{1}{2}}} + O(e^{-cN^2}).$$

Clearly, Propositions 5.1 and 5.2 give Proposition 2.1. First, as the constant c in Proposition 5.2 is independent from δ , we can choose δ small enough so that $O(e^{-cN^2})$ is negligible. Indeed, since $G_{N,M}^{-1}$ is at most of order $e^{C\delta N^2}$, for $\delta \leq c/2C$

$$G_{N,M} e^{-cN^2} \leq e^{-cN^2/2}$$

is negligible compared to

$$G_{N,M} e^{CM^2} e^{-\frac{\beta}{2} NM (\int V(\eta) d\mu_{\text{eq}}(\eta) + \inf \mathcal{J}^{V;B})} \frac{1}{N^{\frac{M}{2}}}$$

which is decaying only polynomially by Proposition 5.1. We get (2.4) by using (5.5), Propositions 5.1 and 5.2, we choose ϵ small enough so that $2\epsilon e^{4C} \leq 1$ so that the terms in e^{CM^2} disappear, and we observe that $(N - M)/N \leq 1$. To derive the lower bound (2.5), we use (5.6) together with Propositions 5.1 and 5.2. in this case the term $((N - 1)/N)^{\frac{\beta}{2}(N-1)}$ is of order one.

The proof of Proposition 5.2 is based on the following proposition.

Proposition 5.3. *Under Assumption 1 and 2, there exists positive constants c, C, δ_0 so that for $\delta \in (0, \delta_0)$, $M \leq \delta N$, for any η_1, \dots, η_M belonging to $[c_0 - \epsilon, c_0 + \epsilon]$, we have the following uniform estimate:*

$$F_{N,M} \Xi(\eta_1, \dots, \eta_M) \lesssim e^{CM^2} e^{-\frac{\beta NM}{2} \int V(\eta) d\mu_{\text{eq}}(\eta)} e^{-\frac{\beta}{2} N \sum_{j=1}^M \mathcal{J}^{V;B}(\eta_j)} + O(e^{-cN^2}). \quad (5.9)$$

Moreover, for $\eta \in [c_0 - \epsilon, c_0 + \epsilon]$

$$F_{N,1} \Xi(\eta) \gtrsim e^{-\frac{\beta N}{2} (\int V(x) d\mu_{\text{eq}}(x) + \mathcal{J}^{V;B}(\eta))} + O(e^{-cN^2}). \quad (5.10)$$

Let us first deduce Proposition 5.2 from Proposition 5.3. The proof is straightforward since by Proposition 5.3

$$\begin{aligned} F_{N,M} L_{N,M} &\lesssim e^{CM^2} e^{-\frac{\beta}{2} NM \int V(\eta) d\mu_{\text{eq}}(\eta) - NM \frac{\beta}{2} \inf \mathcal{J}^{V;B}} \left(\int_{c_0 - \epsilon}^{c_0 + \epsilon} e^{-N \tilde{\mathcal{J}}^{V;B}(x)} dx \right)^M \\ &\quad + O(e^{-cN^2}) \end{aligned}$$

where we can use Laplace method (recall we assume $\tilde{\mathcal{J}}^{V;B}(c_0) = 0$, $\frac{d}{dx} \tilde{\mathcal{J}}^{V;B}(c_0) = 0$, $\frac{d^2}{dx^2} \tilde{\mathcal{J}}^{V;B}(c_0) > 0$, see [AGZ10, section 3.5.3] for details) to get

$$\int_{c_0 - \epsilon}^{c_0 + \epsilon} e^{-N \tilde{\mathcal{J}}^{V;B}(\lambda)} d\lambda \approx \frac{1}{\sqrt{N}}.$$

The proof of the lower bound is similarly deduced from (5.10).

Remark 5.4. Note that if we would have assumed instead of $\frac{d^2}{dx^2} \tilde{\mathcal{J}}^{V;\mathbf{B}}(c_0) > 0$ that for some $q > 2$, $\tilde{\mathcal{J}}^{V;\mathbf{B}}(x) \simeq |x - c_0|^q$ in a neighborhood of c_0 , we would have obtained

$$F_{N,M} L_{N,M} \lesssim C e^{CM^2} e^{-\frac{\beta}{2} NM} \int V(\eta) d\mu_{\text{eq}}(\eta) \frac{1}{N^{\frac{M}{q}}} + O(e^{-cN^2})$$

and criticality would have occurred at $\beta = 2/q$.

5.1. Proof of Proposition 5.1 and 5.3 in the one cut case. To estimate $G_{N,M}$ and $F_{N,M} L_{N,M}$ in the one cut case, we shall rely on the following results from [BG11]:

Theorem 5.5. [BG11, Propositions 1.1 and 1.2] *If V satisfies Assumption 1 and 2 on \mathbf{A} , there exists a universal constant e , and constants $F_\beta^{\{k\}}$ so that we have for N large enough:*

$$Z_{N,\beta}^{V;\mathbf{A}} = N^{(\frac{\beta}{2})N+e} \exp \left(\sum_{k=-2}^K N^{-k} F_\beta^{\{k\}} + o(N^{-K}) \right). \quad (5.11)$$

Moreover, define the correlators given for $x \in \mathbb{C} \setminus \mathbf{A}$ by:

$$W_1(x) := \mu_{N,\beta}^{V;\mathbf{A}} \left(\sum \frac{1}{x - \lambda_i} \right), \quad W_1^{\{-1\}}(x) := \mu_{\text{eq}} \left(\frac{1}{x - \lambda} \right). \quad (5.12)$$

Then

$$W_1(x) = N W_1^{\{-1\}}(x) + O(1). \quad (5.13)$$

(5.13) holds uniformly for x in compact regions outside \mathbf{A} (in particular near the critical point c_0).

Proof of Proposition 5.1. The fact that

$$G_{N,M} \approx C_M e^{MNA_\beta} \frac{1}{N^{\frac{M\beta}{2}}} \left(\frac{N-M}{N} \right)^{\frac{\beta}{2}(N-M)}, \quad (5.14)$$

where $A_\beta = -2F_\beta^{\{-2\}}$ does not depend on N or M and $C_M \lesssim e^{CM^2}$ is a direct consequence of (5.11). Moreover, the large deviation principle of Theorem 1.1(1) yields

$$-F_\beta^{\{-2\}} = \inf \mathcal{E} = \frac{\beta}{2} \left(\int V(x) d\mu_{\text{eq}}(x) - \int \int \log |x - y| d\mu_{\text{eq}}(x) d\mu_{\text{eq}}(y) \right).$$

Also, by definition, since the effective potential is constant on the support of the equilibrium measure,

$$\inf \mathcal{J}^{V;\mathbf{B}} = -C_V = \int V(x) d\mu_{\text{eq}}(x) - 2 \int \int \log |x - y| d\mu_{\text{eq}}(x) d\mu_{\text{eq}}(y).$$

As a consequence

$$A_\beta = \frac{\beta}{2} \inf_{\xi \in \mathbf{A}} \mathcal{J}^{V;\mathbf{A}}(\xi) + \frac{\beta}{2} \int V(\eta) d\mu_{\text{eq}}(\eta).$$

Proof of Proposition 5.3. In the one-cut case we can prove this proposition without the error terms $O(e^{-cN^2})$ which we will need to deal with the several cut case. The main tool is to use concentration of measure. In fact we can write

$$F_{N,M} \Xi(\eta_1, \dots, \eta_M) = \mu_{N-M,\beta}^{V;\mathbf{A}} \left(e^{\sum_{i=1}^{N-M} h_\eta(\lambda_i)} \right)$$

with

$$h_\eta(x) = \beta \sum_{i=1}^M (\ln(\eta_i - x) - \frac{1}{2}V(\eta_i) - \frac{1}{2}V(x)).$$

Eventhough this function depends on M , concentration inequalities will allow a uniform control. We develop the necessary estimates in the appendix, see Lemma A.3. $\|f\|_{\mathcal{L}}$ denotes the Lipschitz norm and $\|f\|_{\infty}$ the uniform bound on a neighborhood of \mathbf{A} . We apply Lemma A.3 with $h = h_\eta$. As the η_i are away from \mathbf{A} , $\|h_\eta\|_{\mathcal{L}}^2$ is uniformly bounded by CM^2 for some finite constant C , whereas $\|h_\eta\|_{\infty}$ is of order M . Hence, we deduce from Lemma A.3 that

$$e^{-CM} e^{(N-M)\mu_{\text{eq}}(h_\eta)} \leq F_{N,M} \Xi(\eta_1, \dots, \eta_M) \leq e^{CM^2} e^{(N-M)\mu_{\text{eq}}(h_\eta)}. \quad (5.15)$$

Note that we can replace above $(N-M)\mu_{\text{eq}}(h_\eta)$ by $N\mu_{\text{eq}}(h_\eta)$ up to an error of order M^2 which amounts to change the constant C . This completes the proof of Proposition 5.3 since

$$\begin{aligned} \mu_{\text{eq}}(h_\eta) &= \frac{\beta}{2} \left(\sum_{i=1}^M 2 \int \log |\eta_i - x| d\mu_{\text{eq}}(x) - V(\eta_i) \right) - \frac{\beta}{2} M \int V(x) d\mu_{\text{eq}}(x) \\ &= -\frac{\beta}{2} \sum_{i=1}^M \mathcal{J}^{V; \mathbf{B}}(\eta_i) - \frac{\beta}{2} M \int V(x) d\mu_{\text{eq}}(x). \end{aligned} \quad (5.16)$$

5.2. Proof of Propositions 5.1 and 5.3 in the general multi-cut case. In the multi cut case, we have to be more careful since the number of eigenvalues that are in each connected component of the support of μ_{eq} is not a priori fixed. The idea is therefore that we will have to sum over all possible number of eigenvalues in these components. Our proof is based on estimates from [BG13] on the fixed filling fraction measure (see Definition 5.6 below). We introduce some new notation below.

From the viewpoint of Large Deviation Principle on L_N , the number of eigenvalues in the interval A_h , which we will call the filling fractions, should in principle be proportionnal to $\mu_{\text{eq}}(S_h)$, which is the mass of equilibrium measure accumulated on A_h .

Thus, let's define \mathbf{f}_\star as the g -tuple denoting the mass of the equilibrium measure in each of the intervals that comprise the support of the equilibrium measure, i.e.

$$\mathbf{f}_\star := (\mu_{\text{eq}}(S_1), \dots, \mu_{\text{eq}}(S_g)) \quad (5.17)$$

In order to describe how the N eigenvalues are distributed in the g intervals \mathbf{A}_h , we let

$$\mathcal{E}_g := \left\{ (f_1, \dots, f_g) \mid \sum_{h=1}^g f_h = 1, f_1, \dots, f_g \geq 0 \right\}. \quad (5.18)$$

Now we can define the fixed filling fraction probability measure:

Definition 5.6. For any $\vec{N} = (N_1, \dots, N_g)$ so that $\vec{N}/N \in \mathcal{E}_g$, let the fixed filling fraction probability measure $d\mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}$ be given by:

$$d\mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}(\lambda) := \frac{1}{Z_{N, \vec{f}, \beta}^{V; \mathbf{A}}} \prod_{h=1}^g \left[\prod_{i=1}^{N_h} d\lambda_{h,i} \mathbf{1}_{\mathbf{A}_h}(\lambda_{h,i}) e^{-\frac{\beta N}{2} V(\lambda_{h,i})} \prod_{1 \leq i < j \leq N} |\lambda_{h,i} - \lambda_{h,j}|^\beta \right] \\ \times \prod_{1 \leq h < h' \leq g} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}}} |\lambda_{h,i} - \lambda_{h',j}|^\beta,$$

where $Z_{N, \vec{f}, \beta}^{V; \mathbf{A}}$ is the partition function.

The following precise estimate of the fixed filling fraction measure from [BG13, Theorem 1.4] will be essential in our proof. It extends Theorem 5.5.

Theorem 5.7. If V satisfies Assumption 1 and 2 on \mathbf{A} , there exists $t > 0$ such that, uniformly for $\vec{N}/N \in \mathcal{E}_g$ that satisfies $|\vec{N}/N - \mathbf{f}_\star| < t$, we have:

$$\frac{N!}{\prod_{h=1}^g (N_h)!} Z_{N, \vec{N}, \beta}^{V; \mathbf{A}} = N^{(\frac{\beta}{2})N+e} \exp \left(\sum_{k=-2}^K N^{-k} F_{\vec{N}/N, \beta}^{\{k\}} + o(N^{-K}) \right). \quad (5.19)$$

e is some universal constant, $F_{\vec{f}, \beta}^{\{k\}}$ extends as a smooth function for \mathbf{f} close enough to \mathbf{f}_\star , and at the value $\mathbf{f} = \mathbf{f}_\star$, the derivative of $F_{\vec{f}, \beta}^{\{-2\}}$ vanishes and its Hessian is negative definite. Assume that \vec{N}/N converges towards \mathbf{f} . Then, the law of the empirical measure L_N under $\mu_{N, \vec{f}, \beta}^{V; \mathbf{A}}$ satisfies a large deviation principle with speed N^2 and good rate function $\tilde{\mathcal{E}}_{\mathbf{f}}$ which is minimized at a unique probability measure $\mu_{\text{eq}, \mathbf{f}}$, which is also the minimum of \mathcal{E} under the constraint that $\mu(\mathbf{A}_h) = f_h$. In particular L_N converges $\mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}$ almost surely to $\mu_{\text{eq}, \mathbf{f}}$. Moreover, for $x \in \mathbb{C} \setminus \mathbf{A}$ let

$$W_{\vec{N}/N}(x) := \mu_{N, \vec{N}, \beta}^{V; \mathbf{A}} \left(\sum \frac{1}{x - \lambda_i} \right), \quad W_{\mathbf{f}}^{\{-1\}}(x) := \mu_{\text{eq}, \mathbf{f}} \left(\frac{1}{x - \lambda} \right). \quad (5.20)$$

Then, there exists $t > 0$ such that, uniformly for $\mathbf{f} \in \mathcal{E}_g$ and $|\mathbf{f} - \mathbf{f}_\star| < t$, we have an expansion for the correlators:

$$W_{\vec{N}/N}(x) = N W_{\vec{N}/N}^{\{-1\}}(x) + O(1). \quad (5.21)$$

(5.21) holds uniformly for x in compact regions outside \mathbf{A} (in our case in particular near the critical point c_0). $\mathbf{f} \rightarrow W_{\mathbf{f}}^{\{-1\}}(x)$ extends as a smooth function in a neighborhood of \mathbf{f}_\star for any $x \in \mathbb{C} \setminus \mathbf{S}$.

Proof of Proposition 5.1. By the partition function estimate from Theorem 5.7, for a sufficiently small κ , we have

$$\begin{aligned} Z_{N-M,\beta}^{V;\mathbf{A}} &= \sum_{N_1+\dots+N_g=N-M} \frac{(N-M)!}{N_1! \dots N_g!} \cdot Z_{N-M, \frac{\vec{N}}{N-M}, \beta}^{V;\mathbf{A}}, \\ &\approx \sum_{\substack{N_1+\dots+N_g=N-M, \\ |\frac{\vec{N}}{N-M} - \mathbf{f}_\star| < \kappa}} (N-M)^{(\frac{g}{2})(N-M)+e} \exp \left((N-M)^2 F_{\frac{\vec{N}}{N-M}, \beta}^{\{-2\}} + (N-M) F_{\frac{\vec{N}}{N-M}, \beta}^{\{-1\}} \right) \\ &\quad + O(e^{-C_\kappa N^2}) Z_{N-M,\beta}^{V;\mathbf{A}}, \end{aligned} \quad (5.22)$$

with some $C_\kappa > 0$. In the last step we applied the large deviation principle for the empirical measure L_{N-M} ; in other words, the sum over all \vec{N} such that $|\frac{\vec{N}}{N-M} - \mathbf{f}_\star| \geq \kappa$ divided by $Z_{N-M,\beta}^{V;\mathbf{A}}$ is negligible since it is the probability that the filling fractions are away from the equilibrium ones, a set on which the distance between the empirical measure L_{N-M} and the equilibrium measure is positive. We used Theorem 5.7 to estimate each partition functions in the remaining sum. Similarly,

$$\begin{aligned} Z_{N,\beta}^{V;\mathbf{A}} &= \sum_{N_1+\dots+N_g=N} \frac{N!}{N_1! \dots N_g!} \cdot Z_{N, \frac{\vec{N}}{N}, \beta}^{V;\mathbf{A}}, \\ &\approx \sum_{\substack{N_1+\dots+N_g=N, \\ |\frac{\vec{N}}{N} - \mathbf{f}_\star| < \kappa}} (N)^{(\frac{g}{2})(N)+e} \exp \left((N)^2 F_{\frac{\vec{N}}{N}, \beta}^{\{-2\}} + (N) F_{\frac{\vec{N}}{N}, \beta}^{\{-1\}} \right). \end{aligned} \quad (5.23)$$

All that is left to do is to analyze the limiting behavior of :

$$L_K := \sum_{\substack{N_1+\dots+N_g=K, \\ |\frac{\vec{N}}{K} - \mathbf{f}_\star| < \kappa}} \exp((K)^2 F_{\frac{\vec{N}}{K}, \beta}^{\{-2\}} + (K) F_{\frac{\vec{N}}{K}, \beta}^{\{-1\}}). \quad (5.24)$$

Here $\vec{N} = (N_1, \dots, N_g)$ with $\sum N_i = K$. This is done in Lemma 5.8 below, from where the rest of the argument is exactly as in the one-cut case:

Lemma 5.8.

$$L_K \approx \exp(K^2 F_{\mathbf{f}_\star, \beta}^{\{-2\}} + K F_{\mathbf{f}_\star, \beta}^{\{-1\}}) \quad (5.25)$$

Proof. According to Theorem 5.7, for \mathbf{f} sufficiently close to \mathbf{f}_\star , $F_{\mathbf{f}}^{\{-1\}}$ and $F_{\mathbf{f}}^{\{-2\}}$ are smooth, and the Hessian of $F_{\mathbf{f}, \beta}^{\{-2\}}$ is negative definite at \mathbf{f}_\star . Thus we can find constant c, C such that for $|\mathbf{f} - \mathbf{f}_\star| \leq \kappa < t$, we have

$$|F_{\mathbf{f}, \beta}^{\{-1\}} - F_{\mathbf{f}_\star, \beta}^{\{-1\}}| \leq C |\mathbf{f} - \mathbf{f}_\star|, \quad (5.26)$$

$$F_{\mathbf{f}, \beta}^{\{-2\}} - F_{\mathbf{f}_\star, \beta}^{\{-2\}} \leq -c |\mathbf{f} - \mathbf{f}_\star|^2, \quad (5.27)$$

$$|F_{\mathbf{f}, \beta}^{\{-2\}} - F_{\mathbf{f}_\star, \beta}^{\{-2\}}| \leq C |\mathbf{f} - \mathbf{f}_\star|^2. \quad (5.28)$$

We first derive the lower bound of L_K . Indeed, there exists at least one $\vec{N}_1 := (N_1, \dots, N_g)$, such that $|\frac{\vec{N}_1}{K} - \mathbf{f}_\star| \lesssim \frac{1}{K}$ and $N_1 + \dots + N_g = K$. Thus we can easily get the following lower bound of L_K with (5.26), (5.28):

$$L_K \geq \exp(K^2 F_{\frac{\vec{N}_1}{K}, \beta}^{\{-2\}} + K F_{\frac{\vec{N}_1}{K}, \beta}^{\{-1\}}) \gtrsim \exp(K^2 F_{\mathbf{f}_\star, \beta}^{\{-2\}} + K F_{\mathbf{f}_\star, \beta}^{\{-1\}}). \quad (5.29)$$

Next, we compute the upper bound of L_K . Direct computation leads to

$$\begin{aligned}
 & \frac{L_K}{\exp(K^2 F_{\mathbf{f}_\star, \beta}^{\{-2\}} + K F_{\mathbf{f}_\star, \beta}^{\{-1\}})} \\
 &= \sum_{\substack{N_1 + \dots + N_g = K, \\ |\frac{\vec{N}}{K} - \mathbf{f}_\star| < \kappa}} \exp(K^2 (F_{\frac{\vec{N}}{K}, \beta}^{\{-2\}} - F_{\mathbf{f}_\star, \beta}^{\{-2\}}) + K (F_{\frac{\vec{N}}{K}, \beta}^{\{-1\}} - F_{\mathbf{f}_\star, \beta}^{\{-1\}})) \\
 &\leq \sum_{|\frac{\vec{N}}{K} - \mathbf{f}_\star| \leq \kappa} \exp(-cK^2 |\frac{\vec{N}}{K} - \mathbf{f}_\star|^2 + CK |\frac{\vec{N}}{K} - \mathbf{f}_\star|).
 \end{aligned} \tag{5.30}$$

In the last step of (5.30), we use (5.26) and (5.27). It is clear that the above right hand side is bounded from which the claim follows. \square

Having established Lemma 5.8, we complete the proof of Proposition 5.1 as in the one-cut case.

To prove Proposition 5.3, let us notice first that we can replace Ξ by taking the integral only on filling fractions close to that of the equilibrium measure since the error will be otherwise of order e^{-cN^2} : for a given configuration $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{N-M-1})$ denote $\vec{N}(\boldsymbol{\lambda}) = (N_1(\lambda), \dots, N_g(\lambda))$ with $N_h(\lambda) = \#\{i : \lambda_i \in \mathbf{A}_h\}$. Then hereafter we replace Ξ by its localized version :

$$\Xi(\eta_1, \dots, \eta_M) := \mu_{N-M, \beta}^{\frac{N-M}{2}V; \mathbf{A}} \left(\prod_{j=1}^M e^{\beta \sum_{i=1}^{N-M} \ln |\eta_j - \lambda_i| - \frac{\beta}{2} (N-M)V(\eta_j)} \mathbf{1}_{|\frac{\vec{N}(\boldsymbol{\lambda})}{N-M} - \mathbf{f}_\star| < \kappa} \right) \tag{5.31}$$

For any $\kappa > 0$, the part we cut-off can be controlled by large deviation principle of the empirical measure L_N , which is of order $e^{-c\kappa N^2}$. All our estimates on $F_{N,M} \Xi$ below will be made up to this error that we will not write down to simplify the exposition. The proof goes again through an expansion of terms where the filling fractions are fixed.

$$F_{N,M} \Xi(\eta_1, \dots, \eta_M) = \sum_{\substack{N-M = N_1 + \dots + N_g, \\ |\frac{\vec{N}}{N-M} - \mathbf{f}_\star| < \kappa}} c_{\vec{N}} d_{\vec{N}}, \tag{5.32}$$

where

$$c_{\vec{N}} := \frac{1}{Z_{N-M, \beta}^{V; \mathbf{A}}} \frac{(N-M)!}{N_1! \dots N_g!} \cdot Z_{N-M, \frac{\vec{N}}{N-M}, \beta}^{V; \mathbf{A}}, \tag{5.33}$$

$$d_{\vec{N}} := \mu_{N-M, \frac{\vec{N}}{N-M}, \beta}^{V; \mathbf{A}} \left(\prod_{i=1}^M e^{\sum_{j=1}^{N-M} (-\frac{\beta}{2} V(\eta_i) + \beta \ln |\lambda_j - \eta_i| - \frac{\beta}{2} V(\lambda_j))} \right). \tag{5.34}$$

We use the concentration Lemma A.3 with

$$h(x) = \sum_{i=1}^M h_{\eta_i}(x), \quad h_{\eta_i}(x) = -\frac{\beta}{2} V(x) + \beta \ln |\eta_i - x|.$$

Note that $\|h\|_{\mathcal{L}}^2$ is of order M^2 for η_1, \dots, η_M close to c_0 and $\|h\|_{\infty}$ is of order M . We first estimate $d_{\vec{N}}$ and then substitute the estimate into (5.32).

$$\begin{aligned}
 d_{\vec{N}} &= \mu_{N-M, \frac{\vec{N}}{N-M}; \beta}^{V; \mathbf{A}} (e^{\sum_{i=1}^{N-M} h(\lambda_i)}) \prod_{j=1}^M e^{-\frac{\beta}{2}(N-M)V(\eta_j)} \\
 &\leq C e^{CM^2} e^{(N-M)\mu_{\text{eq}, \frac{\vec{N}}{N-M}}(h)} \prod_{j=1}^M e^{-\frac{\beta}{2}(N-M)V(\eta_j)}. \tag{5.35}
 \end{aligned}$$

Next, we want to substitute $\mu_{\text{eq}, \frac{\vec{N}}{N-M}}$ by μ_{eq} . By Appendix A.1 in [BG13], $\mathbf{f} \rightarrow \mu_{\text{eq}, \mathbf{f}}(h_\eta)$ is Lipschitz in a neighborhood of \mathbf{f}_\star , uniformly in $\eta \in [c_0 - \epsilon, c_0 + \epsilon]$ so that

$$|\mu_{\text{eq}}(h) - \mu_{\text{eq}, \frac{\vec{N}}{N-M}}(h)| = |\mu_{\text{eq}, \mathbf{f}_\star}(h) - \mu_{\text{eq}, \frac{\vec{N}}{N-M}}(h)| \leq CM \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right|. \tag{5.36}$$

Combining (5.35), (5.32), (5.36) gives

$$\begin{aligned}
 F_{N,M} \Xi(\eta_1, \dots, \eta_M) &\lesssim e^{CM^2} \left(\prod_{j=1}^M e^{-\frac{\beta}{2}NV(\eta_j)} \right) \\
 &\times \sum_{\left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right| < \kappa} c_{\vec{N}} \exp \left((N-M)(\mu_{\text{eq}}^{V; \mathbf{A}}(h) + CM \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right|) + O(M) \right). \tag{5.37}
 \end{aligned}$$

Finally, observe as in (5.30) that $c_{\vec{N}}$ has a sub-Gaussian tail, that is:

$$c_{\vec{N}} \leq C e^{-c(N-M)^2 \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right|^2 + C(N-M) \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right|}. \tag{5.38}$$

so that we deduce that

$$\sum_{\left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right| < \kappa} c_{\vec{N}} e^{CM(N-M) \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right|} \leq C e^{CM^2}, \tag{5.39}$$

Indeed,

$$CM(N-M) \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right| \leq \frac{2}{c} C^2 M^2 + \frac{c}{2} |N-M|^2 \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right|^2$$

so that the term $c_{\vec{N}} e^{CM(N-M) \left| \frac{\vec{N}}{N-M} - \mathbf{f}_\star \right|}$ has also a sub-Gaussian tail up to multiplying it by e^{CM^2} , thus (5.39) follows from (5.38). Therefore (5.37) yields the desired upper bound, as in (5.16):

$$\begin{aligned}
 F_{N,M} \Xi(\eta_1, \dots, \eta_M) &\lesssim e^{CM^2} \left(\prod_{j=1}^M e^{-\frac{\beta}{2}(N-M)V(\eta_j)} \right) e^{(N-M)\mu_{\text{eq}}(h)} \\
 &\leq C e^{CM^2} e^{-\frac{\beta NM}{2} \int V(\eta) d\mu_{\text{eq}}(\eta)} e^{-N \frac{\beta}{2} \sum_{j=1}^M \mathcal{J}^{V; \mathbf{B}}(\eta_j)}. \tag{5.40}
 \end{aligned}$$

The proof of (2.5) is similar to the one cut case. From (5.32), choosing \vec{N} so that $|\vec{N} - N\mathbf{f}_\star| \leq 1$, we get

$$F_{N,1} \Xi(\eta) \geq c_{\vec{N}} d_{\vec{N}}.$$

We use Lemma A.3 to lower bound the term in $d_{\vec{N}}$:

$$d_{\vec{N}} \geq e^{-cM^2} e^{-\frac{\beta}{2}N(\mathcal{J}^{V; \mathbf{B}}(\eta) + \int V(x) d\mu_{\text{eq}}(x))}.$$

The $c_{\vec{N}}$ term is bounded from below by the partition function estimate from Theorem 5.7 as well as the upper bound for $Z_{N,\beta}^{V;\mathbf{A}}$ provided by (5.23) and Lemma 5.8.

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APPENDIX A. CONCENTRATION LEMMAS

A.1. Moments estimate. Here we deduce the following lemma by using the estimate for correlator, by a direct application of Cauchy's integral formula.

Lemma A.1. *Let Assumption 1 and 2 hold, let $\vec{N} = (N_1, \dots, N_g)$ so that $\sum N_i = N$ and let $\mu_{N,\vec{N},\beta}^{V;\mathbf{A}}$ be the fixed filling fractions measure, and $\mu_{\text{eq},\vec{N}}$ be its limiting measure. Let h be a function that is holomorphic in a open neighborhood U of \mathbf{A} . Then,*

$$|\mu_{N,\vec{N},\beta}^{V;\mathbf{A}}(\sum h(\lambda_i)) - N\mu_{\text{eq},\vec{N}}(h)| \lesssim C\|h\|_\infty. \quad (\text{A.1})$$

where $\|h\|_\infty$ is the supremum norm of h on a contour around \mathbf{A} inside U .

This result in particular hold in the one cut case where the fraction is always set to $\vec{N} = N$.

Proof. As h is holomorphic, we can write by Cauchy formula, for a contour \mathcal{C} around \mathbf{A} ,

$$\mu_{N,\vec{N},\beta}^{V;\mathbf{A}}(\sum h(\lambda_i)) = \frac{1}{2i\pi} \int_{\mathcal{C}} h(\xi) W_{\mathbf{f}}(\xi) d\xi$$

from which the estimate follows from (5.21) (and (5.13) in the one cut case). \square

A.2. Concentration estimates. We assumed in Assumption 2 that V is strictly convex in a neighborhood of \mathbf{A} in order to use the following concentration inequality, see sections 2.3.2, 4.4.17, 4.4.26 in [AGZ10] for more details. Note here that we fix the component in which each eigenvalue is living so that indeed they only see a convex potential.

Lemma A.2 (Concentration Inequality). *Let $\mathbf{f} \in \mathcal{E}_g$ be given. Let V be a smooth function such that $V''(x) \geq C > 0$ for all $x \in \mathbf{A}$. Let h be a function that is class C^1 on \mathbb{R}^N . Let $\vec{N} = (N_1, \dots, N_g)$ so that $\sum N_i = N$. Then*

$$\mu_{N,\vec{N},\beta}^{V;\mathbf{A}} \left[\exp \left\{ \left(f - \mu_{N,\vec{N},\beta}^{V;\mathbf{A}}(h) \right) \right\} \right] \lesssim e^{\frac{1}{N^{\frac{1}{\sigma}}} \|h\|_{\mathcal{L}}^2}, \quad (\text{A.2})$$

where

$$\|h\|_{\mathcal{L}} := \sqrt{\sum_{i=1}^N \sup_{x \in \mathbf{A}^N} |\partial_{\lambda_i} h(x)|^2}.$$

Note that this lemma applies in particular in the one cut case.

Lemma A.3. *Let Assumption 1 and 2 hold and let $\vec{N} = (N_1, \dots, N_g)$ so that $\sum N_i = N$. Then, there exists a finite constant C so that for any h holomorphic in an open neighborhood of \mathbf{A} , we have*

$$e^{-C\|h\|_\infty} \lesssim \mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}(\exp(\sum_{i=1}^N (h(\lambda_i) - \int h(\eta) d\mu_{\text{eq}, \vec{N}}(\eta)))) \lesssim e^{C(\|h\|_\mathcal{L}^2 + \|h\|_\infty)}. \quad (\text{A.3})$$

Proof. By Jensen's Inequality:

$$\mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}(\exp(\sum h(\lambda_i))) \geq \exp(\mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}(\sum h(\lambda_i))). \quad (\text{A.4})$$

The upper bound is based on the concentration equality for fixed filling fractions of Lemma A.2 with $\|\sum h(\lambda_i)\|_\mathcal{L}^2 = N\|h\|_\mathcal{L}^2$ so that

$$\mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}(\exp(\sum h(\lambda_i))) \leq \exp\{\frac{1}{C}\|h\|_\mathcal{L}^2\} \exp(\mu_{N, \vec{N}, \beta}^{V; \mathbf{A}}(\sum h(\lambda_i))). \quad (\text{A.5})$$

Lemma A.2 completes the proof. \square

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